Single Variable Expressions for the Efficiency of a Reciprocal Power Transfer System

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Abstract

The analytical expressions for the efficiency of a reciprocal power transfer system as function of multiple parameters, i.e. the elements of its impedance matrix, already exist. In this work, closed expressions for this efficiency as function of a single parameter, i.e. the extended *kQ* factor, are derived. This is done for three representative configurations: (i) maximum efficiency, (ii) maximum power transfer and (iii) conjugate image set-up. The derived formulas are useful for the design and optimization of different types of power transfer systems.

Keywords: *kQ* factor, maximum power transfer efficiency, power conversion efficiency, power transfer, reciprocal circuit, two-port networks.

I. Introduction

We consider a general reciprocal power transfer system as a two-port network with at port #1 a time-harmonic voltage source and at port #2 a passive load Z_L (Fig. 1). The goal of the system is to transfer power from the voltage source to the load Z_L. The power transfer circuit (including the internal impedance of the source) is characterized by its impedance matrix $\mathbf{Z} = \mathbf{R} + j\mathbf{X}$, with elements $z_{ij} = r_{ij} + jx_{ij}$ $(i, j = 1, 2)$. For a linear and reciprocal two-port network ($z_{12} = z_{21}$), the relation between the peak current and peak voltage phasors (as defined in Fig. 1) at the ports is given by:

$$V_1 = z_{11}I_1 + z_{12}I_2 \quad (1)$$

$$V_2 = z_{12}I_1 + z_{22}I_2 \quad (2)$$

Note that we do not specify the circuitry used for the power transfer. The system is considered as a ‘black box’, fully characterized by its impedance matrix $\mathbf{Z}$.

In this paper, we will analyze this reciprocal power transfer system for three representative configurations:

- configuration (i) maximizes the energy transfer efficiency of the system from the source to the load.
- configuration (ii) maximizes the transferred power to the load.
- configuration (iii) applies the conjugate image set-up.

In multiple papers, Ohira [1]–[4] proposed a general extension of the conventional *kQ* product (named the extended *kQ* product α, elaborated further below). He determined an analytical expression for the maximum attainable efficiency of a reciprocal power transfer system, as function of a single parameter: α. This corresponds with configuration (i) from above.

In this work, the method is extended to configurations (ii) and (iii). More specifically, our contributions are:

- An analytical expression is derived for the configurations (ii) and (iii) for the efficiency, as function of the extended *kQ* product α. A simple formula is obtained, valid for *any* general reciprocal power transfer system (section II).

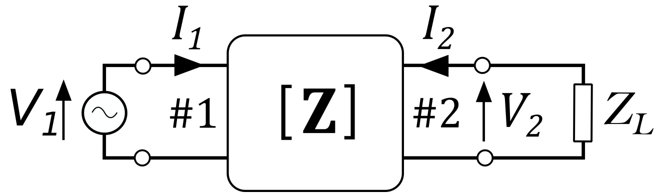

Fig. 1: A general power transfer system as a two-port network with at port #1 a time-sinusoidal voltage source and at port #2 a passive load Z_L.

- We also express our results as function of another parameter, the efficiency angle θ (elaborated further below) (section II).
- We discuss the results and present recommended values for α and θ. Finally, we illustrate for two examples the meaning of the extended *kQ* product α (section III).
- We experimentally validate our results by applying them on an inductive wireless power transfer system (section IV).

II. Derivation of the analytical expressions

A. Maximizing efficiency

We define the efficiency η of a power transfer system as the ratio between the active output power P_{out} delivered to the load relative to the active input power P_{in} provided by the source:

$$\eta = \frac{P_{out}}{P_{in}} \tag{3}$$

The efficiency η will not only be dependent on the impedance matrix **Z** of the two-port network, but also on the applied load Z_L (= $R_L + jX_L$). For obtaining the set-up of configuration (i), we will maximize the efficiency of the power transfer by only changing Z_L. We consider the voltage source and impedance matrix **Z** as given and fixed. In other words, in configuration (i), we will select the specific value for Z_L that maximizes the efficiency η. Notice that this choice of Z_L will usually not maximize the absolute power transfer to the load Z_L. Another value of Z_L, which we will determine in the next section, will be necessary to maximize the power transfer.

By solving the equations

$$\frac{\partial \eta}{\partial R_L} = 0 \tag{4}$$

$$\frac{\partial \eta}{\partial X_L} = 0 \tag{5}$$

the value of the load Z_L for configuration (i) can be determined [5], [6]:

$$R_L = r_{22}\sqrt{1 - \frac{r_{12}^2}{r_{11}r_{22}}}\sqrt{1 + \frac{x_{12}^2}{r_{11}r_{22}}} \tag{6}$$

$$X_L = \frac{r_{12}x_{12}}{r_{11}} - x_{22} \tag{7}$$

With this load at port #2, the maximum attainable efficiency η_{max} can be written as function of the extended *kQ* product α [1]:

$$\eta_{max} = 1 - \frac{2}{1 + \sqrt{1 + \alpha^2}} \tag{8}$$

with the extended *kQ* product α defined as:

$$\alpha = \sqrt{\frac{r_{12}^2 + x_{12}^2}{r_{11}r_{22} - r_{12}^2}} \tag{9}$$

Notice that the maximum efficiency η_{max} is a monotonously increasing function of a single parameter, namely α^2.

Ohira also introduced an angular quantity θ, ranging from 0 to $\pi/4$ rad, which he named the efficiency angle and is defined as [2]–[4]:

$$\tan 2\theta = \alpha \tag{10}$$

With this definition, (8) is reduced to the following compact expression:

$$\eta_{max} = \tan^2 \theta \tag{11}$$

The efficiency angle θ can be used as a figure of merit and design aid for wireless power transfer systems [2]–[4]. We refer to aforementioned references for a more detailed clarification of this parameter.

B. Maximizing power transfer

In a real power transfer system, the goal is often not to maximize the efficiency of the system, but to maximize the transferred power to the load. Constructing the set-up of configuration (ii), which maximizes the power transfer, can be realized by choosing the appropriate (matched) load. We again consider the voltage source and impedance matrix $\mathbf{Z}$ as given and fixed. We will now, just as Ohira did for configuration (i), calculate the efficiency for configuration (ii) as function of α.

First, we determine the Thévenin equivalent of the system, as in [5]. The Thévenin voltage V_{th} can be determined by open circuiting port #2 ($I_2 = 0$):

$$V_{th} = \frac{z_{12}}{z_{11}} V_1 \tag{12}$$

The Thévenin impedance is given by the impedance as seen at port #2 with the voltage source short circuited ($V_1 = 0$):

$$Z_{th} = z_{22} - \frac{z_{12}^2}{z_{11}} \tag{13}$$

Taking into account the maximum power transfer theorem [7], the power transfer to the load is maximized when the load $Z_L = R_L + jX_L$ is the complex conjugate of Z_{th}:

$$R_L = \frac{r_{11}^2 r_{22} - r_{11} r_{12}^2 + r_{11} x_{12}^2 + r_{22} x_{11}^2 - 2 r_{12} x_{12} x_{11}}{r_{11}^2 + x_{11}^2} \tag{14}$$

$$X_L = -\frac{x_{11}^2 x_{22} - x_{11} x_{12}^2 + x_{11} r_{12}^2 + x_{22} r_{11}^2 - 2 x_{12} r_{12} r_{11}}{x_{11}^2 + r_{11}^2} \tag{15}$$

Applying this load to port #2 realizes configuration (ii) which maximizes the power transfer. Notice that R_L equals the negative of X_L, but with the resistances and reactances swapped (see(13)).

Because the reactances of load and the Thévenin impedance cancel each other out, the power factor is one. The active power P_{out} dissipated in the load is then given by

$$P_{out} = \frac{1}{2} \Re\big(Z_L |I_L|^2\big) = \frac{|V_{th}|^2}{8R_L} \tag{16}$$

with I_L the current through the load. With (12), we obtain

$$P_{out} = \frac{|z_{12}|^2 |V_1|^2}{8|z_{11}|^2 R_L} \tag{17}$$

The input impedance Z_{in}, as seen at port #1, is given by:

$$Z_{in} = R_{in} + jX_{in} = \frac{V_1}{I_1} \tag{18}$$

When we terminate port #2 with the load impedance

$$Z_L = R_L + jX_L = -\frac{V_2}{I_2} \tag{19}$$

the input impedance Z_{in} can be written as

$$Z_{in} = \frac{z_{11}(z_{22} + Z_L) - z_{12}^2}{z_{22} + Z_L} \tag{20}$$

when we take into account (1), (2) and (19). The active input power P_{in} delivered by the source can then be written as:

$$P_{in} = \frac{1}{2}\Re\left(\frac{|V_1|^2}{Z_{in}}\right) = \frac{1}{2}\frac{R_{in}|V_1|^2}{R_{in}^2 + X_{in}^2} \tag{21}$$

The efficiency η_{power} of the system for the configuration where the power transfer to the load is maximized, can be calculated as:

$$\eta_{power} = \frac{P_{out}}{P_{in}} = \frac{|z_{12}|^2 (R_{in}^2 + X_{in}^2)}{4|z_{11}|^2 R_L R_{in}} \tag{22}$$

After a simple but elaborate algebraic restatement, combining (9), (14), (20) and (22), the efficiency η_{power} can be written as an elegant and short expression:

$$\eta_{power} = \frac{\alpha^2}{4 + 2\alpha^2} \tag{23}$$

with α given by (9). Notice again that also in the configuration of maximum power transfer, the efficiency η_{power} is a monotonously increasing function of one parameter only, namely α^2.

It is also possible to write this expression as function of the efficiency angle θ. With (10), we obtain

$$\eta_{power} = \frac{\sin^2(2\theta)}{4 - 2\sin^2(2\theta)} \tag{24}$$

C. Conjugate image configuration

We study the conjugate image set-up for our power transfer system since this configuration is meaningful for impedance-matching problems and is particularly suitable for determining the limits of power amplification or loss [8]. In this section, Z^* is the complex conjugate of Z.

The conjugate image configuration can be constructed for any two-port network by connecting specific impedances Z_{c1} and Z_{c2} to port #1 and #2 respectively [8]. The following applies for Z_{c1} and Z_{c2} in order to achieve the conjugate image set-up (Fig. 2):

- If we terminate port #2 with a load Z_{c2}, the impedance as seen into port #1 is Z_{c1}^*.
- If we terminate port #1 with an impedance Z_{c1}, the impedance as seen into port #2 is Z_{c2}^*.

The values of $Z_{c1} = R_{c1} + jX_{c1}$ and $Z_{c2} = R_{c2} + jX_{c2}$ which determine the conjugate image configuration are given by [5], [8]:

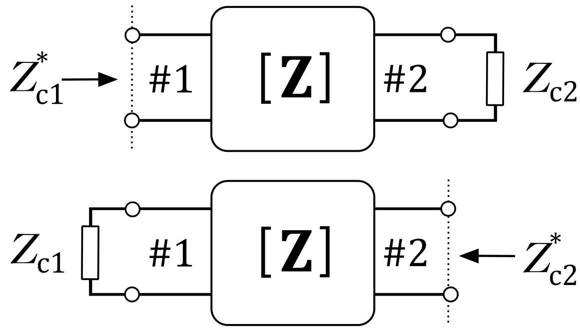

Fig. 2: In the conjugate image configuration, port #1 and #2 of a two-port network are connected to Z_{c1} and Z_{c2} respectively.

$$R_{c1} = r_{11}\sqrt{1 - \frac{r_{12}^2}{r_{11}r_{22}}}\sqrt{1 + \frac{x_{12}^2}{r_{11}r_{22}}} \tag{25}$$

$$X_{c1} = \frac{r_{12}x_{12}}{r_{22}} - x_{11} \tag{26}$$

$$R_{c2} = r_{22}\sqrt{1 - \frac{r_{12}^2}{r_{11}r_{22}}}\sqrt{1 + \frac{x_{12}^2}{r_{11}r_{22}}} \tag{27}$$

$$X_{c2} = \frac{r_{12}x_{12}}{r_{11}} - x_{22} \tag{28}$$

We now calculate the power conversion efficiency for this conjugate image configuration, with the voltage source in series with Z_{c1} at port #1.

Following from the definition of the conjugate image configuration, the power factor is one and the output power P_{out} at the load Z_{c2} is given by:

$$P_{out} = \frac{1}{2}\Re(Z_{c2}|I_2|^2) \tag{29}$$

The input power P_{in} delivered by the supply is

$$P_{in} = \frac{1}{2}\Re(2R_{c1}|I_1|^2) \tag{30}$$

because $2R_{c1}$ is the input impedance as seen from port #1 into the two-port network.

With (2) for the two-port network, the efficiency η_{conj} for the system in conjugate image set-up is given by:

$$\eta_{conj} = \frac{P_{out}}{P_{in}} = \frac{R_{c2}}{2R_{c1}}\left|\frac{z_{12}}{z_{22} + Z_{c2}}\right|^2 \tag{31}$$

After a simple but elaborate algebraic restatement, combining (9), (25), (27) and (32), we obtain:

$$\eta_{conj} = \frac{1}{2}\left(1 - \frac{2}{1 + \sqrt{1 + \alpha^2}}\right) \tag{32}$$

Also in the image conjugate configuration, the efficiency η_{conj} is a monotonously increasing function of one parameter only, namely α^2.

Remark also that that

$$\eta_{conj} = \frac{1}{2}\eta_{max} \tag{33}$$

TABLE I: Analytical expressions for the efficiency η for the three configurations as function of a single parameter, either α or θ.

	$\eta(\alpha)$	$\eta(\theta)$
(i) Maximum efficiency	$1-\frac{2}{1+\sqrt{1+\alpha^2}}$	$\tan^2\theta$
(ii) Maximum power transfer	$\frac{\alpha^2}{4+2\alpha^2}$	$\frac{\sin^2(2\theta)}{4-2\sin^2(2\theta)}$
(iii) Conjugate image	$\frac{1}{2}\left(1-\frac{2}{1+\sqrt{1+\alpha^2}}\right)$	$\frac{1}{2}\tan^2\theta$

This can be understood as follows.

- The values for R_{c2} and X_{c2} which realize the conjugate image set-up are identically the same as the values R_L and X_L which realize the maximum efficiency configuration, as can be seen by (6), (7), (27) and (28).
- The efficiency η_{conj} was calculated with the voltage source in series with Z_{c1} at port #1 and with Z_{c2} considered as the load at port #2. Obviously, considering the definition of the conjugate image set-up, we would obtain the same expression for η_{conj} if we would put the voltage source in series with Z_{c2} at port #2 and consider Z_{c1} as the load at port #1.

Combining the two above observations leads to (33).

Finally, as function of the efficiency angle θ, we can obviously write

$$\eta_{conj} = \frac{1}{2}\tan^2\theta \tag{34}$$

III. Analysis and discussion

A. General analysis and discussion

The analytic expressions for the three different configurations are summarized in Table I. These formulas are valid for all topologies where the relationship between the input and output part can be described by a reciprocal impedance matrix $\mathbf{Z}$, without knowledge of the exact internal circuitry of the power transfer circuit. By measuring or simulating the two-port parameters, one can determine α and calculate the different efficiencies.

Fig. 3 shows the efficiency for the three configurations as function of α^2. Let us first consider the limits:

- If the extended *kQ* factor α goes to zero, the efficiency also goes to zero for all three configurations. This follows from the definition of α (equation (9)). If α approaches zero, the mutual coefficients of the impedance matrix $\mathbf{Z}$, i.e. r_{12} and x_{12}, also approach zero. If both coefficients are zero, this corresponds (see (1) and (2)) with the situation where there is no interaction between port #1 and #2, meaning no power from the source is delivered to the load, leading to an efficiency of zero.
- In the other limit, when the extended *kQ* factor becomes very large, the efficiency in configuration (i) approaches 1. In the ideal case, all the input power from the source can be delivered to the load. In configuration (ii), the efficiency approaches exactly 50%, which is in line with the maximum power transfer theorem [7].

If we would design a power transfer system that obtains 80% of the maximum possible efficiency within each set-up, we would need an α^2 of at least 80, 8 and 80 for configuration (i), (ii) and (iii) respectively.

From Fig. 3, one could mistakenly conclude that a very high extended *kQ* factor leads to a high efficiency *and* a high power transfer. This is not the case. Indeed, one has to remember that the graph on Fig. 3 for configuration (i) is only valid when port #2 is terminated by a load given by (6) and (7).

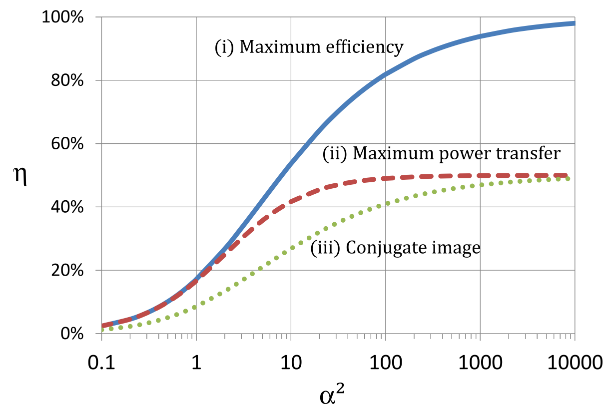

Fig. 3: The efficiency of the three configurations as function of α^2.

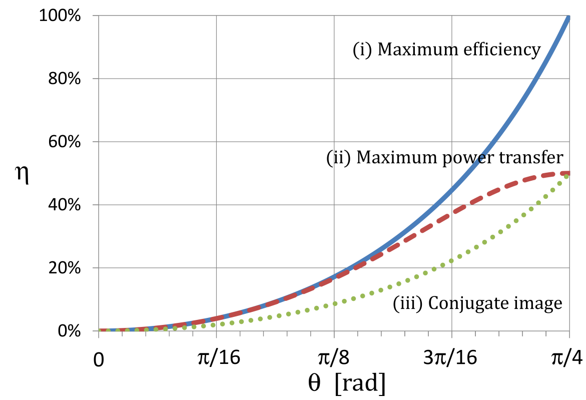

Fig. 4: The efficiency of the three configurations as function of the efficiency angle θ.

Any other load will result in a lower efficiency than η_{max}. In that case, Fig. 3 can be used to compare the set-up with the theoretical maximum for an optimal matched load. In the same way, the graph for configuration (ii) is only valid for a load given by (14) and (15).

Fig. 4 shows the efficiency of the three configurations as function of the efficiency angle θ which, by definition, ranges from 0 to $\pi/4$ rad. In the limits, the same conclusion can be drawn as for the α:

- The efficiency converges to zero when θ approaches zero for all configurations.
- The efficiency is 1 if the efficiency angle θ becomes $\pi/4$ rad for configuration (i).
- For configurations (ii) and (iii), the efficiency is exactly 0.5 when the efficiency angle θ is $\pi/4$ rad.

Values for the efficiency angle θ of at least 0.73, 0.62 and 0.73 rad for configuration (i), (ii) and (iii) respectively are necessary for achieving at least 80% of the maximum possible efficiency within each set-up.

Remark that closed expressions of the efficiency of a reciprocal power transfer system as function of the elements of the impedance matrix already existed and are for example nicely derived by Dionigi et al [5]. The main contribution of this work is that we now express these relations as function of the single parameter α (or alternatively θ). As far as we know, this has not yet been done. Moreover, this parameter, the extended *kQ* factor α, has a physical meaning. It is best known for inductive wireless power transfer applications where it is simply called the *kQ* factor and acts as a figure of merit for the power transfer system. But in this work we have shown that α can be used as a figure of merit for any reciprocal linear power transfer system, and not just for inductive coupling applications. This explains why α is called the *extended kQ* factor. To illustrate the physical meaning of α, we discuss our results for the same two examples as in [1]: wireless power transfer with inductive and with capacitive coupling.

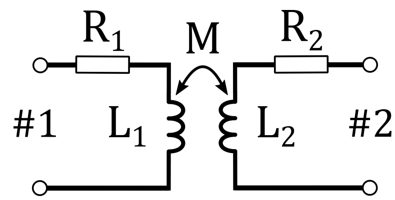

Fig. 5: Two mutually coupled solenoids with inductances L_1 and L_2 and internal resistances R_1 and R_2 respectively. The mutual inductance between the coils is M.

B. Examples

1) Inductive coupling: As first example, we consider a simple power transfer system with inductive coupling, consisting of two mutually coupled solenoids with inductances L_1 and L_2 and internal resistances R_1 and R_2 respectively. The mutual inductance between the coils is M (Fig. 5). The corresponding impedance matrix is given by:

$$\mathbf{Z} = \begin{bmatrix} R_1 + j\omega L_1 & j\omega M \\ j\omega M & R_2 + j\omega L_2 \end{bmatrix} \tag{35}$$

with ω the angular frequency [9]. For this example, the extended *kQ* factor α can be calculated from (9):

$$\alpha = \frac{\omega M}{\sqrt{R_1 R_2}} \tag{36}$$

If we call k the coupling factor between the coils, defined as

$$k = \frac{M}{\sqrt{L_1 L_2}}, \tag{37}$$

and Q_1 and Q_2 the quality factors of each coil respectively, defined as [10]:

$$Q_i = \frac{\omega L_i}{R_i}, \text{with } i = 1, 2 \tag{38}$$

we can write (36) as:

$$\alpha = k\sqrt{Q_1 Q_2} \tag{39}$$

which corresponds with the familiar *kQ* factor for inductive coupling applications.

From (8), the efficiency η_{max} then results in:

$$\eta_{max} = \frac{k^2 Q_1 Q_2}{(1 + \sqrt{1 + k^2 Q_1 Q_2})^2} \tag{40}$$

This equation corresponds with the well-known equation for the maximum efficiency of an inductively coupled system [11].

The efficiency η_{power} in the configuration with maximum power transfer becomes in this example:

$$\eta_{power} = \frac{k^2 Q_1 Q_2}{4 + 2k^2 Q_1 Q_2} \tag{41}$$

and for the conjugate image set-up, we obtain obviously

$$\eta_{conj} = \frac{k^2 Q_1 Q_2}{2(1 + \sqrt{1 + k^2 Q_1 Q_2})^2} \tag{42}$$

Notice that the efficiencies are independent on the values of the inductances L_1 and L_2 of the two coils. The maximum efficiency is only dependent on the mutual inductance M and the internal resistances R_1 and R_2.

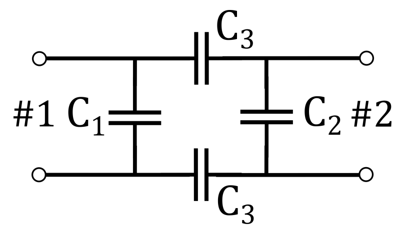

Fig. 6: Wireless power transfer link with capacitive coupling, consisting of two pairs of parallel facing capacitances C_3. The capacitances C_1 and C_2 are parasitic capacitances. Each capacitance has parasitic losses in parallel, not shown on the figure.

This example illustrates the physical meaning of α for inductive coupling: α equals the *kQ* factor of the wireless power transfer system. However, since this parameter is useful and valid for *any* reciprocal power transfer system, and not only for inductive coupling, Ohira named it the extended *kQ* factor [1]. It also explains why Ohira choose α as the single parameter even though the efficiency is a function of α^2.

2) Capacitive coupling: As a second example, we consider another simple power transfer system, now with capacitive coupling, consisting of two pairs of parallel facing capacitances (Fig. 6). The capacitance C_3 is responsible for the wireless link. The capacitances C_1 and C_2 are parasitic capacitances. The capacitances C_1, C_2 and C_3 have parasitic losses, represented by resistances R_1, R_2 and R_3 respectively in parallel with each capacitance (not shown on Fig 6). For this example, the extended kQ-factor α can be calculated from (9) and equals [1]:

$$\alpha = \sqrt{\frac{R_1 R_2}{2R_3} \frac{1 + \omega^2 C_3^2 R_3^2}{R_1 + R_2 + 2R_3}} \tag{43}$$

which for an ideal lossless C_3 ($R_3 = \infty$) leads to

$$\alpha = \frac{1}{2} \omega C_3 \sqrt{R_1 R_2} \tag{44}$$

Notice that this expression is to some extend comparable with the value for α found for inductive coupling and corresponds with the *kQ* factor of this capacitive coupling system.

In this ideal lossless state, the efficiencies for our configurations become:

$$\eta_{max} = \frac{\omega^2 C_3^2 R_1 R_2}{(2 + \sqrt{4 + \omega^2 C_3^2 R_1 R_2})^2} \tag{45}$$

$$\eta_{power} = \frac{\omega^2 C_3^2 R_1 R_2}{16 + 2\omega^2 C_3^2 R_1 R_2} \tag{46}$$

$$\eta_{conj} = \frac{\omega^2 C_3^2 R_1 R_2}{2(2 + \sqrt{4 + \omega^2 C_3^2 R_1 R_2})^2} \tag{47}$$

IV. Experimental validation

We experimentally validate our results by applying them on our example of inductive wireless power transfer (section III.B.1). We set up the wireless link of Fig. 5 by using two planar coils. Each coil has a diameter of 43 mm and consists of two layers of litz wire. The first coil L_1, which is used as transmitter, has 5 turns. The second coil L_2, the receiver, has 10 turns. The coils are centered above each other, with a vertical spacing of 1 mm between them. We measure the inductances L_1 and L_2 of each coil, the mutual inductance M and the series resistance R_1 and R_2 with an Agilent 4285A LCR meter at a frequency of 100 kHz. The results can be found in Table II. Applying equations (37), (38) and (39), the coupling

TABLE II: Measured and calculated values for the inductive wireless power transfer setup.

L_1	10.8 μH $\pm$ 0.05 μH	L_2	43.6 $\pm$ 0.05 μH	M	20.1 μH $\pm$ 0.05 μH
R_1	0.17 Ω $\pm$ 0.005 Ω	R_2	0.80 Ω $\pm$ 0.005 Ω	k	92.42 % $\pm$ 0.27 %
Q_1	40.1 $\pm$ 1.4	Q_2	34.3 $\pm$ 0.3	α	34.3 $\pm$ 0.8

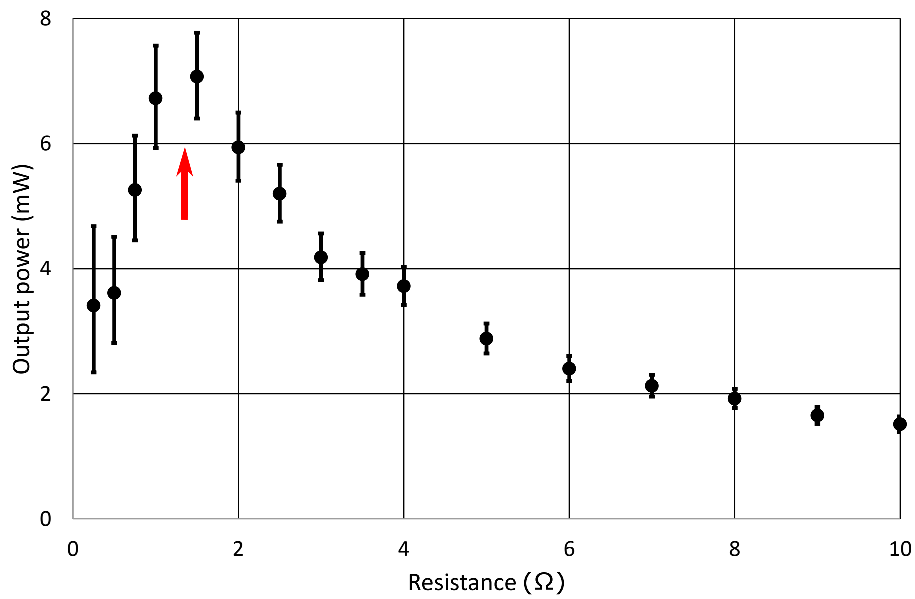

Fig. 7: The measured output power as function of the resistance R_L. The calculated optimal R_L is indicated with an arrow.

factor k, the quality factors Q_1 and Q_2 of each coil, and the extended *kQ* product α are calculated (see Table II).

Ohira [1]–[4] determined an analytical expression for the maximum attainable efficiency of a reciprocal power transfer system, as function of a single parameter: α. This corresponds with configuration (i) we discussed above. In this work, we extended this result to configurations (ii) and (iii). We first start by experimentally validating the setup of configuration (ii), i.e. the configuration that maximizes the transferred power.

We apply a sinusoidal harmonic voltage source at the transmitter side with a peak to peak voltage of V_P =200 mV $\pm$ 5 mV at 100 kHz $\pm$ 0.2 kHz. Given the values of Table II and (35), we can calculate the optimal load with equations (14) and (15). The optimized value of the impedance at 100 kHz results in a resistor of 1.38 Ω $\pm$ 0.03 Ω, in series with a capacitance of 399 nF $\pm$ 26 nF.

For the first measurement, we construct the circuit of configuration (ii): at the receiver side we connect a resistance R_L of 1.40 Ω $\pm$ 0.005 Ω and a capacitance C_L of 396 nF $\pm$ 1 nF. We first vary the resistance R_L when keeping the capacitance C_L fixed. For different values of R_L, we measure the output power dissipated at the resistance R_L (Fig. 7). We notice that the maximum power output is obtained at the optimal resistance. Secondly, we repeat this experiment, but now vary the capacitance C_L when keeping the resistance R_L fixed. The results can be found in Fig. 8 and within the margin of error, the same conclusion can be drawn, confirming the validity of (14) and (15).

We will now for the three configurations analytically calculate and experimentally measure the power transfer efficiency. Given the low values of the optimal resistances, we take for the efficiency measurements the series resistance of the capacitance into account. With the extended *kQ* product α from Table II, we obtain an efficiency η_{power} of 49.91 % $\pm$ 0.004 %, calculated from equation (23). By measuring the input

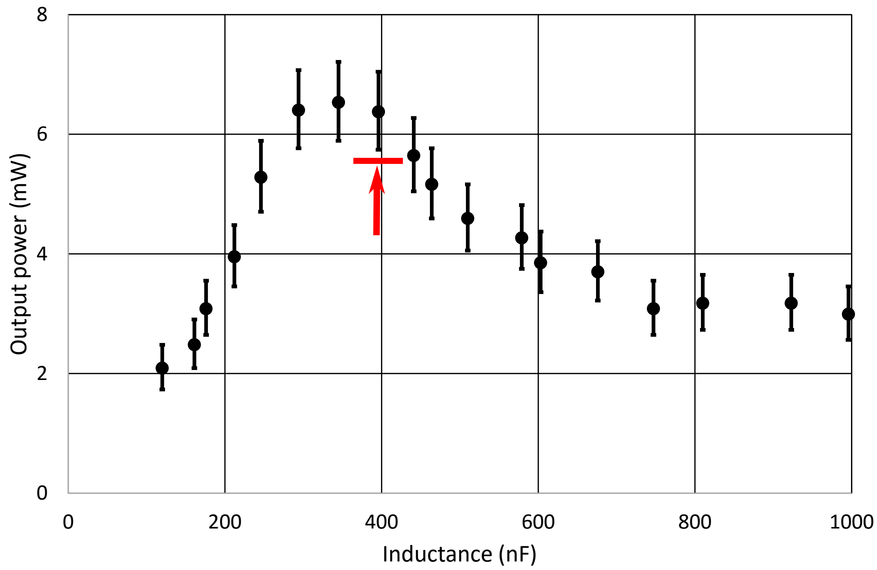

Fig. 8: The measured output power as function of the capacitance C_L. The range of the calculated optimal C_L is given by a horizontal red line, indicated with an arrow.

power when the optimal impedance is applied as load, we obtain a measured efficiency of 47.8 % ± 2.2 %.

We now construct the conjugate image setup of configuration (iii). Given the values of Table II and (35), we can calculate the optimal load with equations (25), (26), (27) and (28). We obtain as optimal impedance at the transmitter side a resistor R_{c1} of 5.82 Ω ± 0.12 Ω in series with a capacitance C_{c1} of 234 nF ± 1 nF. At the receiver side, the optimal impedance consists of a resistor R_{c2} of 27.41 Ω ± 0.56 Ω in series with a capacitance C_{c2} of 58 nF ± 0.5 nF. We construct this setup, i.e. we apply a resistor R_{c1} of 5.80 Ω ± 0.005 Ω in series with a capacitor of 234 nF ± 1 nF at the transmitter side, and a resistor of 27.00 Ω ± 0.005 Ω in series with a capacitor of 58 nF ± 1 nF at the receiver side.

With the calculated extended *kQ* product α from Table II, we can determine the efficiency η_{conj} for this setup, calculated from equation (32). We find a η_{conj} of 47.16 % ± 0.06 %. By measuring the input power delivered by the source and the output power dissipated at the load R_{c2} for this setup, we obtain a measured efficiency of 47.5 % ± 0.4 %.

Since Z_{c2}, the conjugate image impedance at the receiver side, equals the optimal impedance Z_L for configuration (i), we can easily measure the efficiency for the configuration (i) that maximizes the efficiency. We use the same Z_{c2} at the receiver side, but now omit the Z_{c1} at the transmitter side. The theoretical maximum attainable efficiency η_{max}, taken into account the calculated extended *kQ* product α from Table II, can be calculated from (8) and equals 94.3 % ± 0.13 %. As measured efficiency, we find 92.5 % ± 2.1 %. Notice that this measured efficiency in configuration (i) is about the double of the value of the measured efficiency for configuration (iii), as expected. The above experiments confirm the validity of our analytic calculations of section II.

V. Conclusion

We modeled a general reciprocal power transfer system as a two-port network and derived the analytical expressions for the efficiency of the power transfer for three relevant configurations: (i) maximum efficiency, (ii) maximum power transfer and (iii) conjugate image set-up (Table I). The novelty of these expressions lies in the fact that they are dependent of the single parameter α. We also expressed our results as function of the efficiency angle θ. We presented recommended values for α^2 for a general power transfer link of 80, 8 and 80 for configuration (i), (ii) and (iii) respectively and illustrated for two examples the meaning of the extended *kQ* product α. Since the derived formulas are valid for any reciprocal two-port network, we do not even need the concept of, for example, mutual coupling to determine the corresponding extended *kQ* product. Our analytic expressions are useful for the design and optimization of different types of power transfer systems, allowing to better evaluate and compare the performance of any proposed design.

Appendix

As already mentioned, the analytical expressions of the efficiency of a power transfer system as function of the elements of the impedance matrix already existed. The main contribution of this work is that we expressed these equations as function of a single parameter, which has a physical meaning, i.e. the extended *kQ* factor. Earlier formulas of these efficiencies are usually expressed as function of two auxiliary variables ξ and χ, which have no specific physical meaning. They were introduced by Roberts [8] and are defined as:

$$\xi = \frac{r_{12}}{\sqrt{r_{11} r_{22}}} \tag{48}$$

$$\chi = \frac{x_{12}}{\sqrt{r_{11} r_{22}}} \tag{49}$$

Because these auxiliary variables are often reused in the literature (e.g., [5], [12]–[15]), it is relevant to derive the relation between these two auxiliary variables and α. We find:

$$\alpha = \sqrt{\frac{\chi^2 + \xi^2}{1 - \xi^2}} \tag{50}$$

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